

Higher order analogues of unitarity condition for quantum R -matrices

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Abstract

We prove a family of n -th order identities for quantum R -matrices of Baxter-Belavin type in fundamental representation. The set of identities includes the unitarity condition as the simplest one ($n = 2$). Our study is inspired by the fact that the third order identity provides commutativity of the Knizhnik-Zamolodchikov-Bernard connections. On the other hand the same identity gives rise to R -matrix valued Lax pairs for the classical integrable systems of Calogero type. The latter construction uses interpretation of quantum R -matrix as matrix generalization of the Kronecker function. We present a proof of the higher order scalar identities for the Kronecker functions which is then naturally generalized to the R -matrix identities.

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1 Introduction and summary

Quantum $GL(N, \mathbb{C})$ R -matrix is a solution of the quantum Yang-Baxter equation

$$R_{12}^h(z_1, z_2)R_{13}^h(z_1, z_3)R_{23}^h(z_2, z_3) = R_{23}^h(z_2, z_3)R_{13}^h(z_1, z_3)R_{12}^h(z_1, z_2). \quad (1.1)$$

In fundamental representation R -matrix R_{12}^h is an element of $\text{Mat}(N, \mathbb{C})^{\otimes 2}$. R_{ab}^h with $1 \leq a, b \leq n$ is understood as the element of $\text{Mat}(N, \mathbb{C})^{\otimes n}$ which is identical operator in all components of the tensor product except a and b . The projection on a, b -th components coincide with R_{12}^h . The R -matrices under consideration depend on only difference of the spectral parameters. The following notation is used:

$$R_{ab}^h = R_{ab}^h(z_a - z_b). \quad (1.2)$$

We deal with R -matrices which include the rational Yang's one [17]

$$R_{12}^{\text{Yang}}(z_1, z_2) = \frac{1 \otimes 1}{\hbar} + \frac{NP_{12}}{z_1 - z_2}, \quad P_{12} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji} \quad (1.3)$$

as the simplest case, its deformations [6, 15, 9], trigonometric R -matrices [6, 2] and elliptic Baxter-Belavin's R -matrices [3, 5].

Besides (1.1) an R -matrix satisfies also the unitarity condition $R_{12}^h R_{21}^h = 1 \otimes 1$. We write it using different normalization:

$$R_{12}^h R_{21}^h = N^2 \phi(N\hbar, z_1 - z_2) \phi(N\hbar, z_2 - z_1) 1 \otimes 1 = N^2 (\wp(N\hbar) - \wp(z_1 - z_2)) 1 \otimes 1, \quad (1.4)$$

where $\phi(\eta, z)$ is the Kronecker function [16]. Depending on a choice of rational, trigonometric or elliptic case it is equal to¹

$$\phi(\eta, z) = \begin{cases} 1/\eta + 1/z, \\ \coth(\eta) + \coth(z), \\ \frac{\vartheta'(0)\vartheta(\eta+z)}{\vartheta(\eta)\vartheta(z)} \end{cases} \quad (1.5)$$

¹The notations for elliptic functions in this paper coincide with those given in [11] and [12] (see Appendix). One can also find in that papers the definition of the elliptic R -matrix satisfying the properties and identities which are discussed here. Some important definitions and properties are given in this paper as may be necessary.

The r.h.s. of (1.4) contains the following function:

$$\wp(z) = \begin{cases} 1/z^2, \\ 1/\sinh^2(z), \\ \wp(z) - \text{Weierstrass elliptic } \wp\text{-function} \end{cases} \quad (1.6)$$

In (1.4) we have already used the identity

$$\phi(\eta, z)\phi(\eta, -z) = \wp(\eta) - \wp(z). \quad (1.7)$$

It was observed in [13] and later in [10, 11] that the Belavin's R -matrix can be viewed as matrix generalization of the elliptic Kronecker function². In particular, it satisfies the matrix analogue of the Fay identity [7, 16]

$$\phi(\hbar, z)\phi(\eta, w) = \phi(\hbar - \eta, z)\phi(\eta, z + w) + \phi(\eta - \hbar, w)\phi(\hbar, z + w) \quad (1.8)$$

known as the associative Yang-Baxter equation [1]:

$$R_{ac}^{\hbar} R_{cb}^{\eta} = R_{ab}^{\eta} R_{ac}^{\hbar - \eta} + R_{cb}^{\eta - \hbar} R_{ab}^{\hbar}. \quad (1.9)$$

As a consequence of (1.9) together with the unitarity condition (1.4) and the skew-symmetry³

$$R_{ab}^{\hbar}(z_a - z_b) = -R_{ba}^{-\hbar}(z_b - z_a) \quad (1.10)$$

one can derive [10, 12] that R_{ab}^{\hbar} satisfies the Yang-Baxter equation (1.1) and the following cubic identity:

$$\begin{aligned} R_{12}^{\hbar} R_{23}^{\hbar} R_{31}^{\hbar} + R_{13}^{\hbar} R_{32}^{\hbar} R_{21}^{\hbar} &= -N^3 \wp'(N\hbar) 1 \otimes 1 \otimes 1 = \\ &= N^3 \left(\phi(N\hbar, z_{12})\phi(N\hbar, z_{23})\phi(N\hbar, z_{31}) + \phi(N\hbar, z_{13})\phi(N\hbar, z_{32})\phi(N\hbar, z_{21}) \right) 1 \otimes 1 \otimes 1, \end{aligned} \quad (1.11)$$

where $z_{ab} = z_a - z_b$. Some important applications of the latter identity are discussed in the end of the paper.

Purpose of paper is to generalize the unitarity condition (1.4) and identity (1.11) to its higher order analogues. The results are summarized in

Theorem *Let quantum R -matrix satisfies the unitary condition (1.4) and the associative Yang-Baxter equation (1.9). Then it also satisfies the following set of n -th order identities for any $n \in \mathbb{Z}_+$:*

$$\begin{aligned} &\sum_{\substack{1 \leq i_1 \dots i_{n-1} \leq n \\ i_c \neq a; i_b \neq i_c}} R_{ai_1}^{\hbar} R_{i_1 i_2}^{\hbar} \dots R_{i_{n-2} i_{n-1}}^{\hbar} R_{i_{n-1} a}^{\hbar} = \\ &= \underbrace{1 \otimes \dots \otimes 1}_{n \text{ times}} N^n \sum_{\substack{1 \leq i_1 \dots i_{n-1} \leq n \\ i_c \neq a; i_b \neq i_c}} \phi(N\hbar, z_a - z_{i_1}) \phi(N\hbar, z_{i_1} - z_{i_2}) \dots \phi(N\hbar, z_{i_{n-1}} - z_a), \end{aligned} \quad (1.12)$$

²Originally, the idea to consider quantum R -matrix as matrix generalization of a scalar function was proposed in [4]. In that papers the standard normalization of unitarity condition $R_{12}^{\hbar} R_{21}^{\hbar} = 1 \otimes 1$ was used.

³The property (1.10) is the matrix analogue of the Kronecker function property $\phi(\eta, z) = -\phi(-\eta, -z)$. In [13] $R^{\hbar}(z)$ was treated as the classical r -matrix and (1.10) – as the unitarity condition while (1.4) was not used.

where a is a fixed index $1 \leq a \leq n$. For $n \geq 3$ (1.12) can be rewritten as follows:

$$\sum_{\substack{1 \leq i_1 \dots i_{n-1} \leq n \\ i_c \neq a; i_b \neq i_c}} R_{ai_1}^h R_{i_1 i_2}^h \dots R_{i_{n-2} i_{n-1}}^h R_{i_{n-1} a}^h = \underbrace{1 \otimes \dots \otimes 1}_{n \text{ times}} (-N)^n \frac{d^{(n-2)}}{d\eta^{(n-2)}} \wp(\eta) \big|_{\eta=N\hbar} . \quad (1.13)$$

The products of R -matrices in the sums in (1.12), (1.13) contain all possible values of distinct indices from the interval $1, \dots, n$. The total number of terms equals $(n-1)!$. Indeed, since a is fixed the summation index i_1 has $n-1$ possible values. After i_1 is also fixed the next index i_2 has $n-2$ possible values $\{1, \dots, n\} \setminus \{a, i_1\}$, e.t.c. Therefore, the total number of possible sets of indices equals $(n-1)!$. For example, for $n=3$ (1.13) reproduces (1.11), and for $n=4$ it reads as follows:

$$\begin{aligned} & R_{12}^h R_{23}^h R_{34}^h R_{41}^h + R_{12}^h R_{24}^h R_{43}^h R_{31}^h + R_{13}^h R_{32}^h R_{24}^h R_{41}^h + R_{13}^h R_{34}^h R_{42}^h R_{21}^h \\ & + R_{14}^h R_{42}^h R_{23}^h R_{31}^h + R_{14}^h R_{43}^h R_{32}^h R_{21}^h = N^4 \wp''(N\hbar) 1 \otimes 1 \otimes 1 \otimes 1 . \end{aligned} \quad (1.14)$$

The statement of the Theorem implies of course that for $n \geq 3$ the sum of the Kronecker functions products in the r.h.s of (1.12) is equal to $(-1)^n \wp^{(n-2)}(N\hbar)$. This statement looks like it should be well-known. However, we could not find it in the literature⁴. In the next Section we propose a simple proof of this identity in a way which is naturally generalized to R -matrix identities proved in Section 3. The analogue of (1.12) for $n=1$ is discussed in the end of the paper.

2 Higher order identities for Kronecker function

In this Section we will prove

Proposition *The Kronecker function $\phi(\eta, z)$ satisfies the following set of n -th order identities for any $n > 2$:*

$$\sum_{\substack{1 \leq i_1 \dots i_{n-1} \leq n \\ i_c \neq a; i_b \neq i_c}} \phi(\eta, z_a - z_{i_1}) \phi(\eta, z_{i_1} - z_{i_2}) \dots \phi(\eta, z_{i_{n-1}} - z_a) = (-1)^n \frac{d^{(n-2)}}{d\eta^{(n-2)}} \wp(\eta) . \quad (2.1)$$

Before we proceed further let us introduce one more function important for our purposes. It is the first Eisenstein function⁵:

$$E_1(z) = \begin{cases} 1/z , \\ \coth(z) , \\ \wp'(z)/\wp(z) \end{cases} \quad E_1(-z) = -E_1(z) . \quad (2.2)$$

It appears in the expansion of the Kronecker function near the pole $\eta = 0$:

$$\phi(\eta, z) = \eta^{-1} + E_1(z) + \eta (E_1^2(z) - \wp(z))/2 + O(\eta) , \quad (2.3)$$

⁴Presumably, this type of identities can be obtained by combining the elliptic Cauchy determinant formulae proposed by S.N.M. Ruijsenaars [14].

⁵It is simply related to the Weierstrass zeta-function: $E_1(z) = \zeta(z) + \frac{z}{3} \frac{\wp'''(0)}{\wp'(0)}$.

in the formula for derivative

$$\frac{d}{d\eta} \phi(\eta, z) = (E_1(\eta + z) - E_1(\eta))\phi(\eta, z) \quad (2.4)$$

and in degenrated ($\eta = \hbar$) Fay identity (1.8)

$$\phi(\eta, z)\phi(\eta, w) = \phi(\eta, z + w)(E_1(\eta) + E_1(z) + E_1(w) - E_1(z + w + \eta)). \quad (2.5)$$

Combining (2.4) and (2.5) we get

$$\frac{d}{d\eta} \phi(\eta, z) \equiv \phi'(\eta, z) = (E_1(z + y) - E_1(y))\phi(\eta, z) - \phi(\eta, z + y)\phi(\eta, -y). \quad (2.6)$$

Notice that (2.6) is valid for any $y \in \mathbb{C}$ while its l.h.s is independent of y .

A straightforward way for proving of (2.1) type relations is to compare the poles and residues of both sides. It is easy to show that the l.h.s is a double-periodic function of η and z_1, \dots, z_n . Moreover, one can verify that the residues at $z_i = z_j$ equal zero. However, this only means that the l.h.s is a double-periodic function of η which behaves as $1/\eta^n$ near $\eta = 0$. In this way it is hard to fix it, i.e. to prove the absence (in the r.h.s.) of terms $c_k(\tau) \frac{d^k}{d\eta^k} \wp(\eta)$ with $k < n - 2$. Instead, we suggest another proof of (2.1) based on (2.6).

Proof of Proposition:

The proof of (2.1) is by induction on n . For $n = 3$

$$\phi(\eta, z_1 - z_2)\phi(\eta, z_2 - z_3)\phi(\eta, z_3 - z_1) + \phi(\eta, z_1 - z_3)\phi(\eta, z_3 - z_2)\phi(\eta, z_2 - z_1) = -\wp'(\eta)$$

it is well-known. On one can verify it directly by comparing the structure of poles and residues. Alternatively, one can use (2.5) and then (1.7) to rewritten it in a more recognizable form:

$$E_1(\eta + z) + E_1(\eta - z) - 2E_1(\eta) = \frac{\wp'(\eta)}{\wp(\eta) - \wp(z)}, \quad z = z_a - z_b.$$

Let (2.1) is true for n . Differentiate its both sides with respect to η . The r.h.s. is then equal to

$$-(-1)^{n+1} \frac{d^{(n-1)}}{d\eta^{(n-1)}} \wp(\eta), \quad (2.7)$$

that is the r.h.s. of (2.1) taken for $n := n + 1$ with the opposite sign.

For the l.h.s we have:

$$\sum_{\substack{1 \leq i_1 \dots i_{n-1} \leq n \\ i_c \neq a; \ i_b \neq i_c}} \left(\phi'_{ai_1}(\eta)\phi_{i_1i_2}(\eta)\dots\phi_{i_{n-1}a}(\eta) + \dots + \phi_{ai_1}(\eta)\phi_{i_1i_2}(\eta)\dots\phi'_{i_{n-1}a}(\eta) \right), \quad (2.8)$$

where the short notations $\phi_{ij}(\eta) = \phi(\eta, z_i - z_j)$ and $\phi'_{ij}(\eta) = \frac{d}{d\eta} \phi(\eta, z_i - z_j)$ are used.

Substitute (2.6) into (2.8) choosing y each time as follows: for $\phi'_{ij}(\eta)$ let

$$y = z_j - z_{n+1}. \quad (2.9)$$

The answer for (2.8) after the substitution consists of terms of two types:

1. of the form $\phi^n E_1$;
2. of the form ϕ^{n+1} .

The first type terms cancel out for each expression inside the brackets in (2.8) due to skew-symmetry (2.2) of E_1 function. Indeed, summing up all such terms with fixed values of i_1, \dots, i_{n-1} we obtain:

$$0 = \phi_{ai_1}(\eta)\phi_{i_1i_2}(\eta)\dots\phi_{i_{n-1}a}(\eta)\left(E_1(z_a - z_{i_1} + z_{i_1} - z_{n+1}) - E_1(z_{i_1} - z_{n+1}) + \right. \quad (2.10)$$

$$\left. E_1(z_{i_1} - z_{i_2} + z_{i_2} - z_{n+1}) - E_1(z_{i_2} - z_{n+1}) + \dots + E_1(z_{i_{n-1}} - z_a + z_a - z_{n+1}) - E_1(z_a - z_{n+1})\right).$$

The total sum of the second type terms (of from ϕ^{n+1}) is exactly the l.h.s. of (2.1) taken for $n := n + 1$ with the common opposite sign:

$$\sum_{\substack{1 \leq i_1 \dots i_{n-1} \leq n \\ i_c \neq a; i_b \neq i_c}} \left(\phi'_{ai_1}(\eta)\phi_{i_1i_2}(\eta)\dots\phi_{i_{n-1}a}(\eta) + \dots + \phi_{ai_1}(\eta)\phi_{i_1i_2}(\eta)\dots\phi'_{i_{n-1}a}(\eta) \right) =$$

$$- \sum_{\substack{1 \leq i_1 \dots i_{n-1} \leq n \\ i_c \neq a; i_b \neq i_c}} \left(\phi_{a,n+1}(\eta)\phi_{n+1,i_1}(\eta)\phi_{i_1i_2}(\eta)\dots\phi_{i_{n-1}a}(\eta) + \dots \right. \quad (2.11)$$

$$\left. + \phi_{ai_1}(\eta)\phi_{i_1i_2}(\eta)\dots\phi_{i_{n-1},n+1}(\eta)\phi_{n+1,a}(\eta) \right) = - \sum_{\substack{1 \leq i_1 \dots i_n \leq n \\ i_c \neq a; i_b \neq i_c}} \phi_{ai_1}(\eta)\phi_{i_1i_2}(\eta)\dots\phi_{i_na}(\eta).$$

The last equality is easily verified by considering the structure of summation indices. All of them are distinct for each term and differ from the "outer" one a .⁶ The same expression is obtained from the second type terms after the substitution. At the same time the total number of the second type terms in (2.8) (and therefore in (2.11)) equals $n!$ (before differentiating with respect to η we had $(n-1)!$ terms⁷ and then applied the Leibniz rule to each term) which coincide with the number of terms in (2.1) taken for $n := n + 1$.

Formula (2.11) together with (2.7) finishes the inductive proof. ■

3 Higher order R-matrix identities

In this Section we prove the Theorem (1.12), (1.13). The proof is similar to the one of the Proposition from the previous Section.

The R -matrix analogue of the first Eisenstein function (2.2) is the classical r -matrix $r_{ab}(z)$. It appears in the classical limit in the same way as E_1 in (2.3):

$$R_{12}^{\hbar}(z) = \frac{1}{\hbar} 1 \otimes 1 + r_{12}(z) + \hbar m_{12}(z) + O(\hbar). \quad (3.1)$$

⁶The value of the "outer" index a is in fact not important here: see remark (4.1).

⁷See the comment after (1.13).

Similarly to E_1 -function the classical r -matrix is skew-symmetric⁸:

$$r_{ab}(z_a - z_b) = -r_{ba}(z_b - z_a), \quad m_{ab} = m_{ba} = \frac{1}{2} (r_{ab}^2 - 1 \otimes 1 N^2 \wp(z_a - z_b)). \quad (3.2)$$

The R -matrix analogue of (2.6) comes from the associative Yang-Baxter equation (1.9) and (3.1) in the limit $\eta \rightarrow \hbar$:

$$\partial_{\hbar} R_{ab}^{\hbar} \equiv J_{ab}^{\hbar} = R_{ab}^{\hbar} r_{ac} + r_{cb} R_{ab}^{\hbar} - R_{ac}^{\hbar} R_{cb}^{\hbar}. \quad (3.3)$$

Proof of Theorem (1.12), (1.13):

The statement (1.12) for $n = 2$ is the unitarity condition (1.4), for $n = 3$ – it is the known from [10, 12] identity (1.11), and for $n = 1$ it is due to (4.2). Let us prove (1.13). Then (1.12) follows from the scalar analogue of (1.13) – Proposition (2.1).

The idea of the proof repeats the one given for (2.1). The proof of (1.13) is by induction on n . Let it is true for n . Multiply both sides of (1.13) by $\otimes 1$ (1 is identity $N \times N$ matrix) and differentiate them with respect to \hbar . Then the r.h.s. becomes equal to

$$- \underbrace{1 \otimes \dots \otimes 1}_{n+1 \text{ times}} (-N)^{n+1} \frac{d^{(n-1)}}{d\eta^{(n-1)}} \wp(\eta) \big|_{\eta=N\hbar}, \quad (3.4)$$

that is the r.h.s. of (1.13) taken for $n := n + 1$ with the opposite sign.

The l.h.s becomes

$$\sum_{\substack{1 \leq i_1 \dots i_{n-1} \leq n \\ i_c \neq a; i_b \neq i_c}} \left(J_{ai_1}^{\hbar} R_{i_1 i_2}^{\hbar} \dots R_{i_{n-2} i_{n-1}}^{\hbar} R_{i_{n-1} a}^{\hbar} + \dots + R_{ai_1}^{\hbar} R_{i_1 i_2}^{\hbar} \dots R_{i_{n-2} i_{n-1}}^{\hbar} J_{i_{n-1} a}^{\hbar} \right). \quad (3.5)$$

The latter is analogues to (2.8). As in the scalar case, plugging J_{ab}^{\hbar} from (3.3) with $c = n + 1$ we obtain the l.h.s. of (1.13) taken for $n := n + 1$ with the common opposite sign. This answer comes from $-R_{i,n+1}^{\hbar} R_{n+1,j}^{\hbar}$ type terms in (3.3) as in the proof of the scalar identities. Let us only comment the cancellation of terms containing classical r -matrices r_{ab} . The input of such terms into the expression inside the brackets in (3.5) equals

$$\begin{aligned} & (R_{ai_1}^{\hbar} r_{a,n+1} + r_{n+1,i_1} R_{ai_1}^{\hbar}) R_{i_1 i_2}^{\hbar} \dots R_{i_{n-1} a}^{\hbar} + R_{ai_1}^{\hbar} (R_{i_1 i_2}^{\hbar} r_{i_1, n+1} + r_{n+1, i_2} R_{i_1 i_2}^{\hbar}) R_{i_2 i_3}^{\hbar} \dots R_{i_{n-1} a}^{\hbar} + \\ & \dots + R_{ai_1}^{\hbar} R_{i_1 i_2}^{\hbar} \dots R_{i_{n-2} i_{n-1}}^{\hbar} (R_{i_{n-1} a}^{\hbar} r_{i_{n-1}, n+1} + r_{n+1, a} R_{i_{n-1} a}^{\hbar}) \end{aligned} \quad (3.6)$$

Two terms containing $r_{a,n+1}$ are the first and the last ones. They are cancelled because $r_{a,n+1}$ in the first term can be moved⁹ to the last but one position and due to skew-symmetry (3.2). All other terms are combined into the commutator

$$[r_{n+1, i_1} + r_{n+1, i_2} + \dots + r_{n+1, i_{n-1}}, R_{ai_1}^{\hbar} R_{i_1 i_2}^{\hbar} \dots R_{i_{n-1} a}^{\hbar}]$$

⁸The latter follows from either unitarity condition (in \hbar^{-1} , \hbar^0 orders of expansion) or from the skew-symmetry (1.10) (in \hbar^0 , \hbar^1 orders of expansion).

⁹The product $R_{i_1 i_2}^{\hbar} \dots R_{i_{n-2} i_{n-1}}^{\hbar}$ does not contain indices a or $n + 1$, and therefore commutes with $r_{a,n+1}$.

because all r -matrices with indices $r_{n+1,k}$ can be moved to the left positions in the corresponding products, and r -matrices with indices $r_{k,n+1}$ can be moved to the right position in their products. The obtained commutator equals zero since the product of R -matrices is a scalar operator by the inductive assumption. Therefore, expression (3.6) equals zero as well. ■

In the end of the Section let us also write down (1.13) for $n = 5$:

$$\begin{aligned}
& R_{15}^h R_{54}^h R_{43}^h R_{32}^h R_{21}^h + R_{14}^h R_{45}^h R_{53}^h R_{32}^h R_{21}^h + R_{13}^h R_{35}^h R_{54}^h R_{42}^h R_{21}^h + R_{15}^h R_{53}^h R_{34}^h R_{42}^h R_{21}^h \\
& + R_{13}^h R_{34}^h R_{45}^h R_{52}^h R_{21}^h + R_{14}^h R_{43}^h R_{35}^h R_{52}^h R_{21}^h + R_{12}^h R_{25}^h R_{54}^h R_{43}^h R_{31}^h + R_{12}^h R_{24}^h R_{45}^h R_{53}^h R_{31}^h \\
& + R_{15}^h R_{54}^h R_{42}^h R_{23}^h R_{31}^h + R_{14}^h R_{42}^h R_{25}^h R_{53}^h R_{31}^h + R_{14}^h R_{45}^h R_{52}^h R_{23}^h R_{31}^h + R_{15}^h R_{52}^h R_{24}^h R_{43}^h R_{31}^h \\
& + R_{12}^h R_{23}^h R_{35}^h R_{54}^h R_{41}^h + R_{12}^h R_{25}^h R_{53}^h R_{34}^h R_{41}^h + R_{13}^h R_{32}^h R_{25}^h R_{54}^h R_{41}^h + R_{15}^h R_{53}^h R_{32}^h R_{24}^h R_{41}^h \\
& + R_{13}^h R_{35}^h R_{52}^h R_{24}^h R_{41}^h + R_{15}^h R_{52}^h R_{23}^h R_{34}^h R_{41}^h + R_{12}^h R_{23}^h R_{34}^h R_{45}^h R_{51}^h + R_{12}^h R_{24}^h R_{43}^h R_{35}^h R_{51}^h \\
& + R_{13}^h R_{32}^h R_{24}^h R_{45}^h R_{51}^h + R_{14}^h R_{43}^h R_{32}^h R_{25}^h R_{51}^h + R_{13}^h R_{34}^h R_{42}^h R_{25}^h R_{51}^h + R_{14}^h R_{42}^h R_{23}^h R_{35}^h R_{51}^h \\
& = -N^5 \wp'''(N\hbar) 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1.
\end{aligned} \tag{3.7}$$

4 Applications and remarks

- The permutation group of the set (z_1, \dots, z_n) transforms an R -matrix identities (1.12)-(1.13) with some "outer" index a to identity of the same form for different $\tilde{a} \neq a$. One should rename a set of $\{z_k\}$ and act on (1.12)-(1.13) by the corresponding product of permutation operators (from both sides). In the scalar case the identity (2.1) is invariant with respect to the action of the permutation group: its l.h.s is in fact independent of index a because one can rearrange the products of functions ϕ in a way that the total sum acquires the form of the same identity for a different value of the index $\tilde{a} \neq a$. For example, for $n = 3$ obviously

$$\begin{aligned}
& \phi(\eta, z_1 - z_2) \phi(\eta, z_2 - z_3) \phi(\eta, z_3 - z_1) + \phi(\eta, z_1 - z_3) \phi(\eta, z_3 - z_2) \phi(\eta, z_2 - z_1) = \\
& \phi(\eta, z_2 - z_1) \phi(\eta, z_1 - z_3) \phi(\eta, z_3 - z_2) + \phi(\eta, z_2 - z_3) \phi(\eta, z_3 - z_1) \phi(\eta, z_1 - z_2)
\end{aligned} \tag{4.1}$$

Here $a = 1$ and $\tilde{a} = 2$. The same holds true for any n, a, \tilde{a} .

- The R -matrix identity (1.12) makes some sense even for $n = 1$: for $R_{ab}^h(z)$ we can identify the a -th and b -th tensor components keeping $z \neq 0$. Then

$$R_{aa}^h(z) = 1_a N \phi(N\hbar, \frac{z}{N}), \tag{4.2}$$

where 1_a is the identity $N \times N$ matrix in the a -th component. To prove (4.2) consider the most general - elliptic Belavin's R -matrix [5]. In our notations and with our normalization it acquires the form:

$$R_{12}^h(z) = \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_\alpha(z, \omega_\alpha + \hbar) T_\alpha \otimes T_{-\alpha} \in \text{Mat}(N, \mathbb{C})^{\otimes 2}, \tag{4.3}$$

where

$$\varphi_\alpha(z, \omega_\alpha + \hbar) = \exp(2\pi i \frac{\alpha_2}{N} z) \phi(z, \omega_\alpha + \hbar), \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}, \quad (4.4)$$

while $\{T_\alpha\}$ is a special basis in $\text{Mat}(N, \mathbb{C})$:

$$T_\alpha T_\beta = \kappa_{\alpha, \beta} T_{\alpha + \beta}, \quad \kappa_{\alpha, \beta} = \exp\left(\frac{\pi i}{N}(\beta_1 \alpha_2 - \beta_2 \alpha_1)\right), \quad (4.5)$$

i.e.

$$T_\alpha T_{-\alpha} = 1_{N \times N}. \quad (4.6)$$

See details in e.g. Appendix of [12]. From (4.3) and (4.6) we conclude that

$$R_{11}^\hbar(z) = 1_{N \times N} \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_\alpha(z, \omega_\alpha + \hbar). \quad (4.7)$$

The latter sum in the l.h.s. is equal to $N\phi(N\hbar, z/N)$. This is particular case of the finite Fourier transformation formulae for the Kronecker function [11].

- The quantum R -matrices can be used for construction of the classical Lax pairs for the Calogero type models [10]. The \mathfrak{gl}_n Lax matrix is of the form:

$$\mathcal{L}(\hbar) = \sum_{a,b=1}^n \tilde{E}_{ab} \otimes \mathcal{L}_{ab}(\hbar), \quad \mathcal{L}_{ab}(\hbar) = \delta_{ab} p_a 1_a \otimes 1_b + \nu(1 - \delta_{ab}) R_{ab}^\hbar(z_a - z_b), \quad (4.8)$$

where \tilde{E}_{ab} is the standard basis in $\text{Mat}(n, \mathbb{C})$, the set of R -matrix spectral parameters $\{z_a\}$ – are the Calogero particles coordinates, p_a – particles momenta, and \hbar plays the role of the spectral parameter. Formula (4.8) generalizes the Krichever's answer [8]. The following guess was made in [10]: the diagonal elements (blocks) of $\text{tr} \mathcal{L}^k(\hbar)$ are the scalar operators with the coefficient equals to the same element computed for $\text{tr} l(\hbar)^k$ – the (usual) Krichever's Lax matrix $l_{ab}(\hbar) = p_a \delta_{ab} + \nu(1 - \delta_{ab}) N \phi(N\hbar, z_a - z_b)$. For $n = 2$ this statement follows from the unitarity condition (1.4) and for $n = 3$ it follows from the cubic identity (1.11). Obviously the statement of the Theorem (1.12) is of the same type. In fact, (1.12) can be considered as a part of the guess for $k = n$.

- It follows from the quantum Yang-Baxter equation (1.1) and the cubic identity (1.11) that

$$[r_{ab}, m_{ac} + m_{bc}] + [r_{ac}, m_{ab} + m_{bc}] = 0, \quad (4.9)$$

where r_{ab} and m_{ab} are defined from the classical limit expansion (3.1) (see details in [10]). The latter equation guarantees the commutativity of different type KZB connections

$$\begin{aligned} \nabla_a &= \partial_{z_a} + \sum_{b: b \neq a} r_{ab}(z_a - z_b), \quad a = 1, \dots, n; \\ \nabla_\tau &= \partial_\tau + \sum_{b > c} m_{bc}(z_b - z_c), \end{aligned} \quad (4.10)$$

i.e. $[\nabla_a, \nabla_\tau] = 0$ (the commutativity $[\nabla_a, \nabla_b] = 0$ follows from the classical Yang-Baxter equation). The obtained identities (1.12), (1.13) provide a set of equations for r_{ab} , m_{ab} and higher coefficients of the expansion (3.1). The simplest one is the $1/\hbar^{n-2}$ order terms in (1.13):

$$\sum_{a,b,c: c < a < b} [r_{ca}, r_{ab}]_+ + [r_{ab}, r_{bc}]_+ + [r_{bc}, r_{ca}]_+ = -(n-2) \sum_{b,c: b \neq c} m_{bc}. \quad (4.11)$$

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